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Acoustic radiation force on subwavelength objects due to progressive waves

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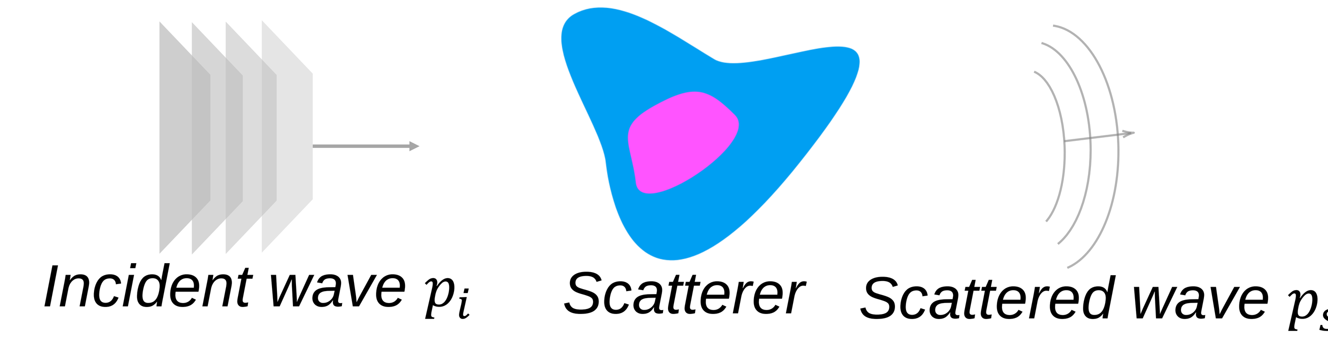


Acoustic radiation force due to progressive waves

- Time-average force $\mathbf{F}$ due to pressure waves $p_i = p_0 e^{i(\mathbf{k}_i \cdot \mathbf{r} - \omega t)}$
- Nonlinear due to quadratic terms in conservation equations
- $\mathbf{F}$ is approximated by calculating scattered wave p_s according to subwavelength limit of **linear acoustic scattering theory**
- Verified by **partial wave expansions** and **Fourier transforms**

Wave equation: $\Box^2 p = f_1(\mathbf{r}) c_0^{-2} \ddot{p} + \nabla \cdot \{3 f_2(\mathbf{r}) / [2 + f_2(\mathbf{r})] \nabla p\}$

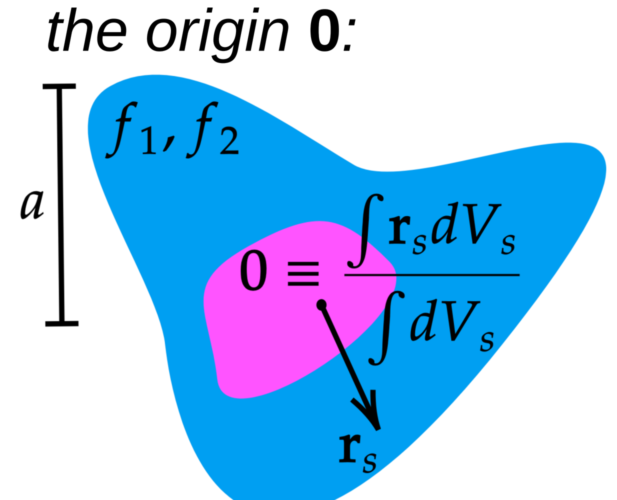
- $f_1(\mathbf{r}) = 1 - \beta(\mathbf{r})/\beta_0$ and $f_2(\mathbf{r}) = 2[\rho(\mathbf{r}) - \rho_0]/[2\rho(\mathbf{r}) + \rho_0]$,[◊] where β_0, ρ_0 = ambient compressibility, density; $c_0^2 = (\beta_0 \rho_0)^{-1}$
- Wave eq. solved implicitly by Helmholtz-Kirchhoff (H-K) integral
- Three approximations reduce H-K integral to Eqs. 1 and 2:
  1. **Far field**: radius (r) $\gg$ scatterer size (a)
  2. **Born**: scattered field (p_s) $\ll$ incident field (p_i)
  3. **Long wavelength**: $a \ll \lambda = 2\pi/k$ = wavelength
- Eqs. 1 and 2 are inserted in $F_\parallel = \langle I \rangle c_0^{-1} \oint |\Phi_s|^2 (1 - \cos\psi) d\Omega$,[†] where $\langle I \rangle = p_0^2 (2\rho_0 c_0)^{-1}$, Ω = solid angle, and $\mathbf{e}_i \cdot \mathbf{e}_r = \cos\psi$
- $\boldsymbol{\alpha}_\mathrm{c} = \mathbf{0}$ for symmetric scatterers and/or for the static limit $k \to 0$

Eqs. 1: Monopole, dipole, and coupling polarizabilities

$$\alpha_\mathrm{m} = -\int_{V_s} f_1(\mathbf{r}_s)\, dV_s, \quad \alpha_\mathrm{d} = \int_{V_s} \frac{3 f_2(\mathbf{r}_s)}{2 + f_2(\mathbf{r}_s)} dV_s,$$

$$\boldsymbol{\alpha}_\mathrm{c} = -k \left[\int_{V_s} f_1(\mathbf{r}_s) \mathbf{r}_s\, dV_s - \cos\psi \int_{V_s} \frac{3 f_2(\mathbf{r}_s)}{2 + f_2(\mathbf{r}_s)} \mathbf{r}_s\, dV_s \right]$$

The ***centroid*** of the scatterer defines the origin **0**:

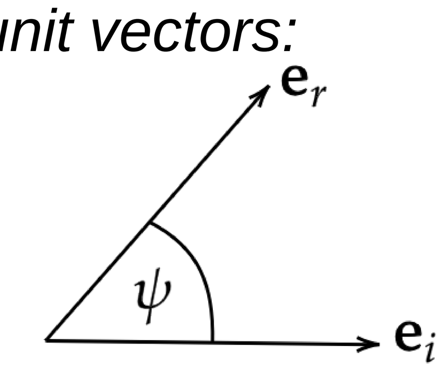


Eq. 2: Directivity Φ and scattered wave p_s

$$\Phi_s(\mathbf{k}_s) = \frac{k^2}{4\pi} [\alpha_\mathrm{m} + i \boldsymbol{\alpha}_\mathrm{c} \cdot (\mathbf{e}_i - \mathbf{e}_r) + \alpha_\mathrm{d} \mathbf{e}_i \cdot \mathbf{e}_r]$$

$$p_s = p_0 \frac{e^{ikr}}{r} \Phi_s(\mathbf{k}_s) \quad \text{Spherically spreading wave}$$

*$\mathbf{e}_i$ and $\mathbf{e}_r$ are the **incident** and **radial** unit vectors:*



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Ex. 1: Spherically symmetric nucleated cell

- $\boldsymbol{\alpha}_\mathrm{c} = \mathbf{0}$; denoting $\chi' \equiv (a'/a)^3$, $\chi'' \equiv (a''/a)^3$,

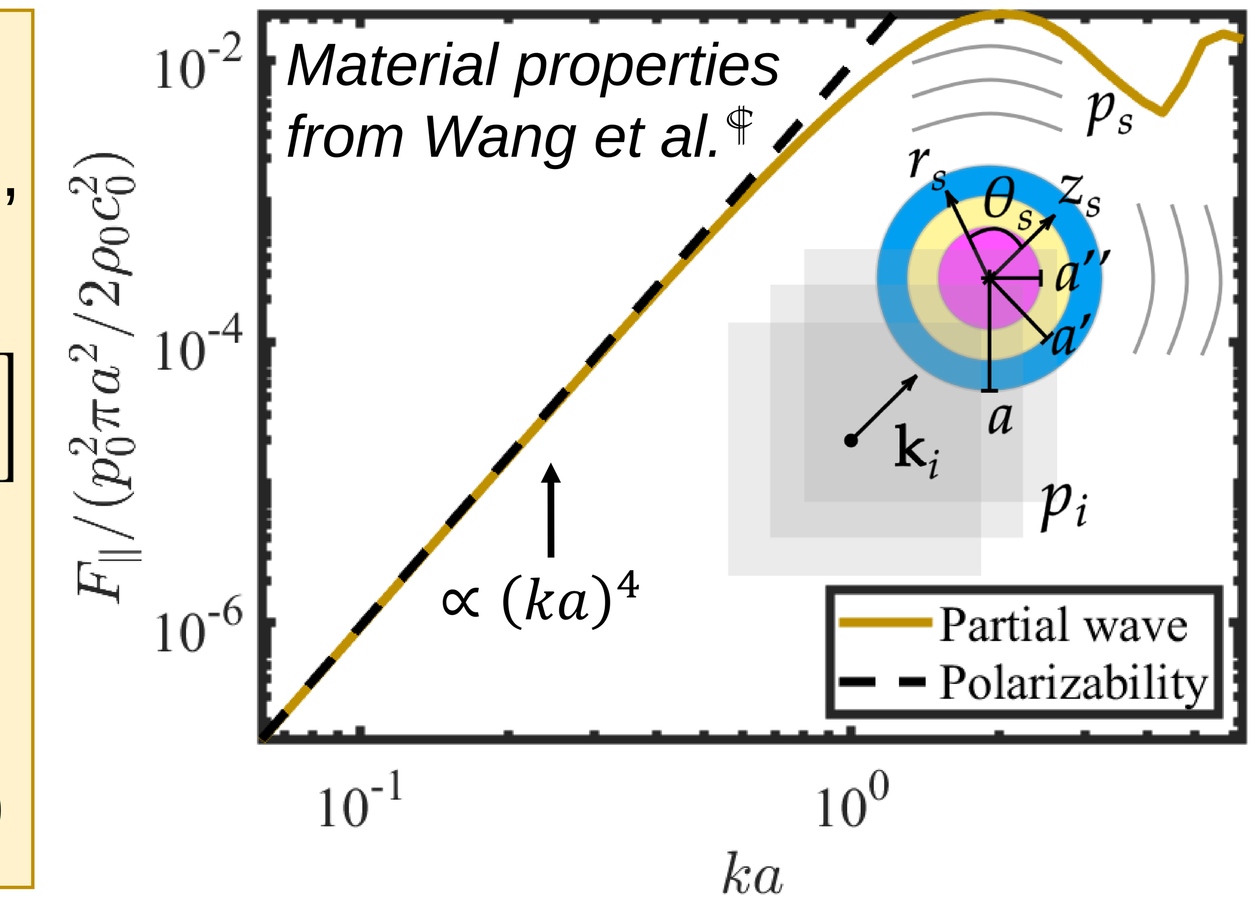
$$\alpha_\mathrm{m} = -\frac{4\pi a^3}{3} [f_1'' \chi'' + f_1'(\chi' - \chi'') + f_1(1 - \chi')],$$

$$\alpha_\mathrm{d} = \frac{4\pi a^3}{3} \left[\frac{3 f_2''/2}{1 + f_2''/2} \chi'' + \frac{3 f_2'/2}{1 + f_2'/2} (\chi' - \chi'') + \frac{3 f_2/2}{1 + f_2/2} (1 - \chi') \right]$$

- Force in direction of incident wave is

$$F_\parallel = \frac{p_0^2}{2\rho_0 c_0^2} \frac{k^4}{4\pi} \left(\alpha_\mathrm{m}^2 - \frac{2}{3} \alpha_\mathrm{m} \alpha_\mathrm{d} + \frac{1}{3} \alpha_\mathrm{d}^2 \right)$$

- $F_\parallel$ is verified by comparison to Eq. 3;[‡] $F_\perp = 0$



Ex. 2: Cube with antisymmetric properties

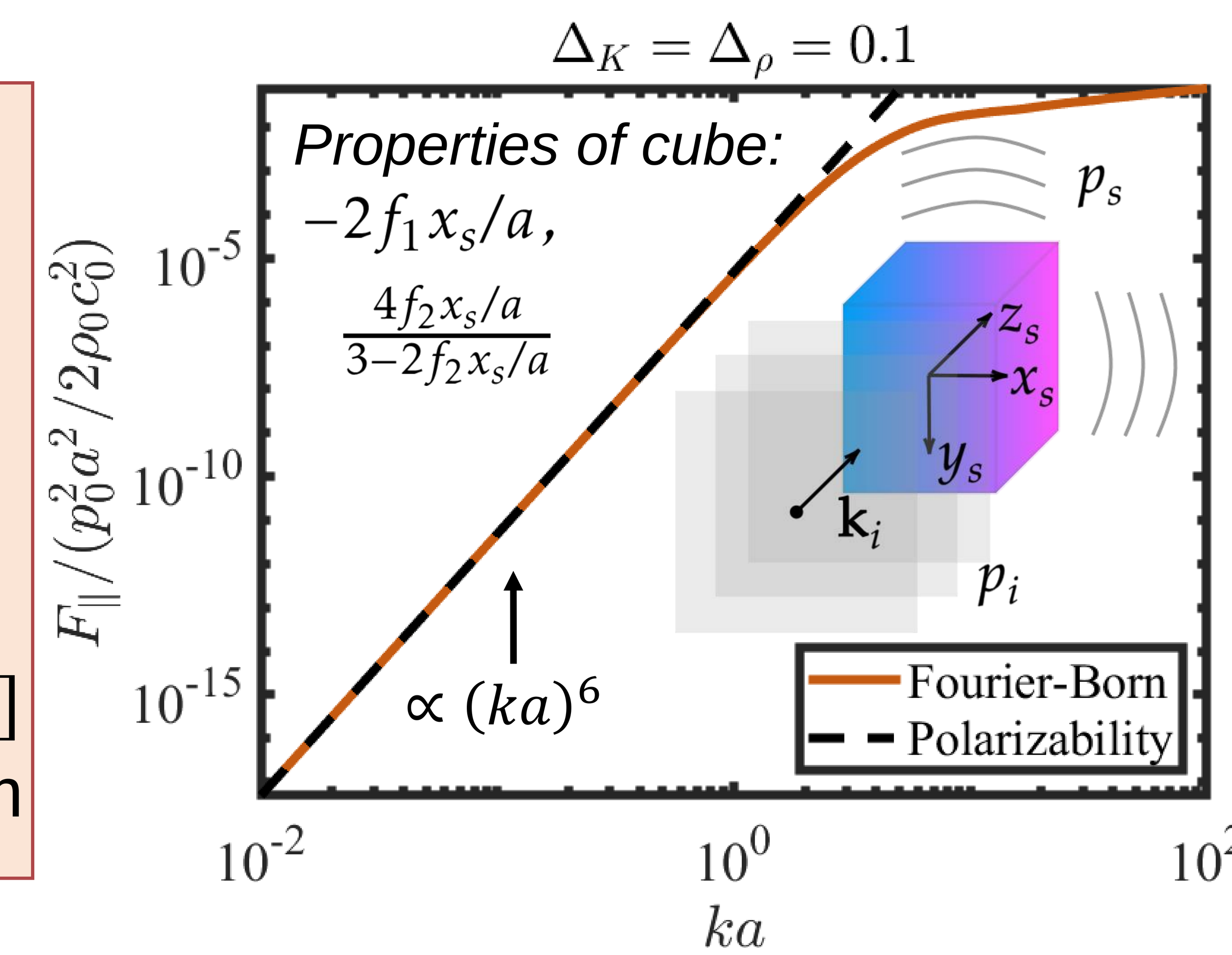
- $\alpha_\mathrm{m} = \alpha_\mathrm{d} = 0$ due to antisymmetry, while

$$\boldsymbol{\alpha}_\mathrm{c} = \frac{ka^4}{6} (f_1 + f_2 \cos\theta) \mathbf{e}_x, \quad \cos\theta = \mathbf{e}_z \cdot \mathbf{e}_r$$

- Force in direction of incident wave is

$$F_\parallel = \frac{(ka)^6}{144\pi} \frac{p_0^2 a^2}{2\rho_0 c_0^2} \left[\frac{1}{3} f_1^2 + \frac{1}{15} (f_2^2 - 2 f_1 f_2) \right]$$

- $F_\perp = -\langle I \rangle c_0^{-1} \oint |\Phi_s|^2 \, \mathbf{e}_x \cdot \mathbf{e}_r \, d\Omega$,[†] $\to 0$, $O[(ka)^6]$
- Verified by Eq. 4, $\mathcal{F}_\mathrm{3D}$ = 3D Fourier transform



Eq. 3: Exact radiation force on layered sphere[*]

$$F_\parallel = \frac{i\pi}{\rho_0 c_0^2 k^2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(n+m+1)(n+m)!}{(2n+1)(2n+3)(n-m)!} \times (A_n^* + A_{n+1} + 2 A_n^* A_{n+1}) a_n^{m*} a_{n+1}^m + \text{c.c.}$$

Eq. 4: Fourier-Born scattering[**]

$$\Phi_s = -\frac{k^2}{4\pi} \left[\mathcal{F}_\mathrm{3D} \left\{ f_1(\mathbf{r}_s) e^{i \mathbf{k}_i \cdot \mathbf{r}_s} \right\} - \cos(\psi) \mathcal{F}_\mathrm{3D} \left\{ \frac{3 f_2(\mathbf{r}_s)}{2 + f_2(\mathbf{r}_s)} e^{i \mathbf{k}_i \cdot \mathbf{r}_s} \right\} \right]$$

Conclusion and references

- Calculated $\mathbf{F} = O[(ka)^6]$ for $ka \ll 1$; verified results analytically and numerically
- Future work: calculate radiation torque due to plane progressive waves

[◊] T. S. Jerome et al., *JASA* **145**, 36-44 (2019); $\Delta_K \equiv (K - K_0)/K_0$, $\Delta_\rho \equiv (\rho - \rho_0)/\rho_0$, where $K = 1/\beta$, $K_0 = 1/\beta_0$.
[†] P. J. Westervelt, *J. Acoust. Soc. Am.* **29**, 26-29 (1957), Eqs. (2) and (16).
[‡] L. P. Gor'kov, *Sov. Phys. Dokl.* **6**, 773-775 (1962), Eq. (10) recovered for homogeneous sphere ($a'' = a' = a$).
[¶] Y.-Y. Wang et al., *J. Appl. Phys.* **122**, 1-6 (2017), Table I.
[*] C. A. Gokani et al., *Proc. Mtgs. Acoust.* **48**, 1-9 (2022), Eq. (10).
[**] P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw-Hill, 1968), Eq. (8.1.20).