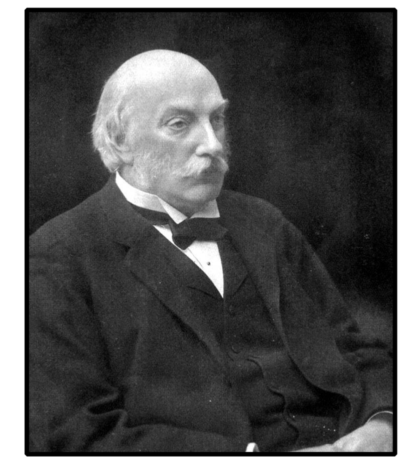
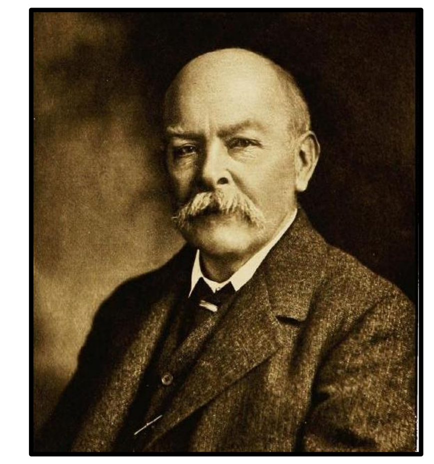
Langevin radiation force and its connection with linear acoustics

Chirag A. Gokani,* Michael R. Haberman, Mark F. Hamilton
Walker Department of Mechanical Engineering and Applied Research Laboratories at University of Texas at Austin

Motivation to study Langevin radiation force

- Steady forces exerted by sound involve quadratic stresses. *Are second-order equations therefore required to calculate the force?*
- Lord Rayleigh and John Henry Poynting debated this question.◊ Poynting's result (formalized by Paul Langevin) neglects the quadratic term in the equation of state, which Rayleigh included.†
- Rayleigh's result applies to constrained fluids, like plane waves in tubes. The Langevin force applies to unconstrained fluids and is of greater relevance to acoustofluidics, acoustic tweezers, etc.
- Nevertheless, traditional derivations of the Langevin force begin with the exact momentum equation, obscuring its linear origins.
- The **Langevin force** is derived below from **linear acoustics**.


Lord Rayleigh


J. H. Poynting

Lord Rayleigh kindly pulled me out of the pit into which I fell, pointing out that when we take into account second-order quantities the ordinary sound equation does not hold. —J. H. Poynting◊

Acoustic wave variables

ρ' = acoustic density $[\mathrm{kg/m^3}]$

$\mathbf{u}$ = fluid velocity vector $[\mathrm{m/s}]$

p = acoustic pressure $[\mathrm{kg/m\,s^2}]$

Ambient properties of the fluid

ρ_0 = ambient density $[\mathrm{kg/m^3}]$

β_0 = compressibility $[\mathrm{m\,s^2/kg}]$

$c_0 = (\beta_0\rho_0)^{-\frac{1}{2}}$ = sound speed $[\mathrm{m/s}]$

Quadratic quantities formed from Eqs. (1)–(3)

$\mathbf{g} = \mathbf{I}/c_0^2 =$ momentum density (8)

$L = \frac{1}{2}\rho_0 u^2 - \frac{1}{2}\beta_0 p^2 =$ Lagrangian density (9)

$\mathsf{S} = \mathsf{I}L - \rho_0 \mathbf{u}\otimes\mathbf{u} =$ radiation stress (10)

Equations of linear acoustics

$$\frac{\partial\rho'}{\partial t} + \rho_0\nabla\cdot\mathbf{u} = 0 \quad \text{conservation of mass} \quad (1)$$

$$\rho' - p/c_0^2 = 0 \quad \text{equation of state} \quad (2)$$

$$\nabla p + \rho_0\frac{\partial\mathbf{u}}{\partial t} = \mathbf{0} \quad \text{conservation of momentum} \quad (3)$$

Solutions of this system are solutions of the wave equation, $\nabla^2 p - c_0^{-2}\ddot{p} = 0$.

Multiplying Eq. (1) by $\mathbf{u}$, using Eq. (2) to eliminate ρ', invoking **Identity A**, and identifying $\mathbf{I} = p\mathbf{u}$ = intensity yields

$$\frac{1}{c_0^2}\frac{\partial\mathbf{I}}{\partial t} - \frac{p}{c_0^2}\frac{\partial\mathbf{u}}{\partial t} + \rho_0[\nabla\cdot(\mathbf{u}\otimes\mathbf{u}) - (\nabla\mathbf{u})\cdot\mathbf{u}] = \mathbf{0}. \quad (4)$$

Multiplying Eq. (3) by p shows that $p\partial\mathbf{u}/\partial t$ in Eq. (4) equals $\nabla(p^2)/2\rho_0$, while **Identity B** shows that $(\nabla\mathbf{u})\cdot\mathbf{u} = \nabla(u^2)/2$.

*Supported by the Chester M. McKinney Graduate Fellowship in Acoustics at ARL:UT

Equation (4) therefore becomes

$$\frac{1}{c_0^2}\frac{\partial\mathbf{I}}{\partial t} + \nabla\left(\tfrac{1}{2}\beta_0 p^2 - \tfrac{1}{2}\rho_0 u^2\right) + \rho_0\nabla\cdot(\mathbf{u}\otimes\mathbf{u}) = \mathbf{0}. \quad (5)$$

The spatial derivatives in Eq. (5) can be recast by noting from **Identity C** that $\nabla(p^2) = \mathsf{I}\cdot\nabla(p^2) = \nabla\cdot(\mathsf{I}p^2) - p^2\nabla\cdot\mathsf{I}$, where I is the identity tensor. Since $\nabla\cdot\mathsf{I} = \mathbf{0}$, $\nabla(p^2)$ and $\nabla(u^2)$ equal $\nabla\cdot(\mathsf{I}p^2)$ and $\nabla\cdot(\mathsf{I}u^2)$, respectively.

Equation (5) therefore becomes

$$\nabla\cdot\left[\mathsf{I}\left(\tfrac{1}{2}\rho_0 u^2 - \tfrac{1}{2}\beta_0 p^2\right) - \rho_0\mathbf{u}\otimes\mathbf{u}\right] = \frac{1}{c_0^2}\frac{\partial\mathbf{I}}{\partial t}. \quad (6)$$

$$\nabla\cdot\mathsf{S} = \frac{\partial\mathbf{g}}{\partial t} \quad (7‡)$$

Langevin radiation force F

For time-harmonic fields, the time average $\langle\cdots\rangle$ of Eq. (7‡) integrated over the volume of a scatterer equals

$$\mathbf{F} = \oint_A \langle\mathsf{S}\rangle\cdot d\mathbf{A}. \quad (11)$$

The Langevin force is analogous to forces exerted by other linear wave fields, like electromagnetic waves and waves on a string.

Vector and tensor calculus identities

A. For two differentiable vector fields $\mathbf{v}$ and $\mathbf{w}$, $\nabla\cdot(\mathbf{v}\otimes\mathbf{w}) = (\nabla\mathbf{v})\cdot\mathbf{w} + (\nabla\cdot\mathbf{w})\cdot\mathbf{v}$, where the outer product $\otimes$ is defined by $(\mathbf{v}\otimes\mathbf{w})\cdot\mathbf{x} = (\mathbf{w}\cdot\mathbf{x})\mathbf{v}$.

B. For $\mathbf{v}$ and $\mathbf{w}$, $\nabla(\mathbf{v}\cdot\mathbf{w}) = (\nabla\mathbf{v})^{\mathrm{T}}\cdot\mathbf{w} + (\nabla\mathbf{w})^{\mathrm{T}}\cdot\mathbf{v}$, where "T" denotes the transpose operation. For $\mathbf{v} = \mathbf{w} = \mathbf{u}$, $\nabla(u^2) = 2(\nabla\mathbf{u})^{\mathrm{T}}\cdot\mathbf{u}$, where $u^2 \equiv \mathbf{u}\cdot\mathbf{u}$. Taking the curl of Eq. (3) shows that $\nabla\times\mathbf{u} = \mathbf{0}$; therefore $(\nabla\mathbf{u})^{\mathrm{T}} = \nabla\mathbf{u}$. Thus $(\nabla\mathbf{u})\cdot\mathbf{u} = \nabla(u^2)/2$.

C. For a differentiable scalar f and rank-2 tensor T, $\nabla\cdot(\mathsf{T}f) = \mathsf{T}\cdot\nabla f + f\nabla\cdot\mathsf{T}$.

Conclusion and references

- Proved that "for the calculation of the [Langevin] force correct up to terms of the second order… it is sufficient to find the solution of the linear scattering problem"**
- Avoided complications encountered in traditional derivations, like distinctions between Eulerian and Lagrangian coordinates and Taylor expansion in the enthalpy

◊ J. H. Poynting, *Philos. Mag.* **9**, 393–406 (1905).
† R. T. Beyer. *J. Acoust. Soc. Am.*, **63**, 1025–1030, 1978.
‡ P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw-Hill, 1968), Eq. (6.2.17).
** L. P. Gor'kov, *Sov. Phys. Dokl.* **6**, 773-775 (1962).