We want to integrate

$$\iiint_{\text{sphere of radius } a} \nabla^2 \left( \frac{e^{ikR}}{4\pi R} \right) \mathrm{d}V \tag{1}$$

Since equation (1) contains derivatives that blow up at $R = 0$, the volume integral must be evaluated as the sum of two indefinite integrals:

$$\begin{aligned} \iiint_{\text{sphere of radius } a} \nabla^2 \left( \frac{e^{ikR}}{4\pi R} \right) \mathrm{d}V = \lim_{\eta \to 0^+} \int_\eta^a \int_0^\pi \int_0^{2\pi} \nabla^2 \left( \frac{e^{ikR}}{4\pi R} \right) R^2 \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}R \\ + \int_0^{0^+} \int_0^\pi \int_0^{2\pi} \nabla^2 \left( \frac{e^{ikR}}{4\pi R} \right) R^2 \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}R \end{aligned}$$

The first integral on the right-hand-side is

$$\begin{aligned} \int_\eta^a \int_0^\pi \int_0^{2\pi} \nabla^2 \left( \frac{e^{ikR}}{4\pi R} \right) R^2 \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}R &= \lim_{\eta \to 0^+} (ikr - 1) e^{ikr} \Big|_\eta^a \\ &= (ika - 1) e^{ika} + 1 \quad \text{(first integral)} \end{aligned}$$

The second integral on the right-hand-side evaluated from $R = 0$ to $R = 0^+$, so the small-argument expansion of the integrand is taken.

$$\begin{aligned} &\int_0^{0^+} \int_0^\pi \int_0^{2\pi} \nabla^2 \left( \frac{1 + ikR - k^2R^2/2! - \ldots}{4\pi R} \right) R^2 \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}R \\ &\simeq \frac{1}{4\pi} \int_0^{0^+} \int_0^\pi \int_0^{2\pi} \nabla^2 \left( \frac{1}{R} \right) \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta \, \mathrm{d}R \qquad \text{(for } R \to 0\text{)} \end{aligned}$$

In this simplification, the Laplacian of the quantity in parentheses goes as $R^{-3}$, which dominates the $R^2$ from the Jacobian, so the $R^2$ vanishes when taking the small-argument.

Now employing Griffiths equation 102 (towards the end of the relevant pages I sent you)

$$\nabla^2 \frac{1}{R} = -4\pi \delta^3(\boldsymbol{R}), \tag{Griffiths 102}$$

The $R$ integral in the equation (for $R \to 0$) can be changed from $\int_0^{0^+}$ to $\int_0^\infty$ because the integrand is now non-zero from 0 to $0^+$, and 0 everywhere else. The second integral on the right-hand-side becomes

$$\frac{1}{4\pi}\iiint_{\text{all space}} -4\pi\delta^3(\boldsymbol{R})\,\mathrm{d}V = -\iiint_{\text{all space}} \delta^3(\boldsymbol{R})\,\mathrm{d}V$$
$$= -1 \qquad \text{(second integral)}$$

Adding the (first integral) and (second integral) gives the integral of equation (1):

$$(ika - 1)e^{ika} + 1 - 1 = (ika - 1)e^{ika}$$

This matches the result of applying the divergence theorem to equation (1).