Systems and Transforms with Applications in Optics

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Table 1-1 Fourier transform theorems

$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega$	$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt$
$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{\omega}{a}\right)$
$f^*(t)$	$F^*(-\omega)$
$F(t)$	$2\pi f(-\omega)$
$f(t - t_o)$	$F(\omega) e^{-jt_o\omega}$
$f(t) e^{j\omega_o t}$	$F(\omega - \omega_o)$
$f(t) \cos \omega_o t$	$\frac{1}{2} [F(\omega + \omega_o) + F(\omega - \omega_o)]$
$f(t) \sin \omega_o t$	$\frac{j}{2} [F(\omega + \omega_o) - F(\omega - \omega_o)]$
$\frac{d^n f(t)}{dt^n}$	$(j\omega)^n F(\omega)$
$(-jt)^n f(t)$	$\frac{d^n F(\omega)}{d\omega^n}$
$m_n = \int_{-\infty}^{\infty} t^n f(t) \, dt$	$F(\omega) = \sum_{n=0}^{\infty} \frac{m_n}{n!} (-j\omega)^n$
$\int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau) \, d\tau$	$F_1(\omega) F_2(\omega)$
$\int_{-\infty}^{\infty} f(t + \tau) f^*(\tau) \, d\tau$	$\lvert F(\omega) \rvert^2$
$\int_{-\infty}^{\infty} \lvert f(t) \rvert^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert F(\omega) \rvert^2 \, d\omega$	
$f(t) + j\hat{f}(t)$	$2F(\omega) U(\omega)$
$\hat{f}(t)$	$-j \operatorname{sgn} \omega F(\omega)$
$\sum_{n=-\infty}^{\infty} f(t + nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{T}\right) e^{j2\pi nt/T}$	

Table 1-2 Examples of Fourier transforms

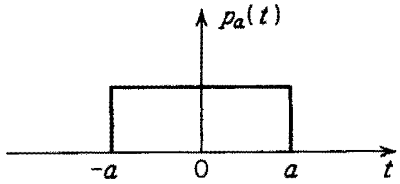
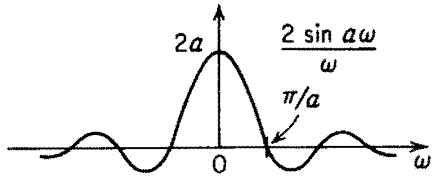
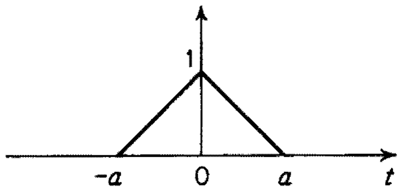
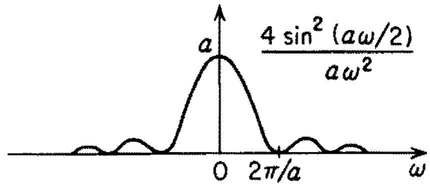
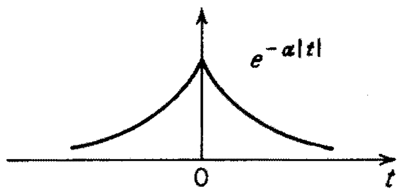
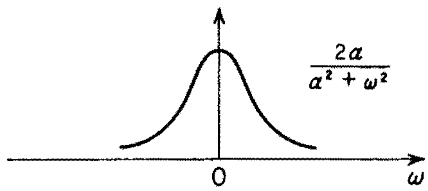
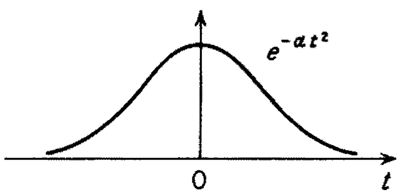
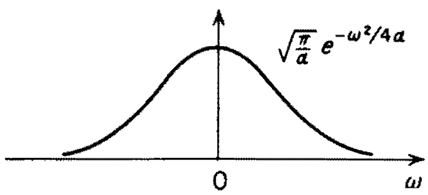
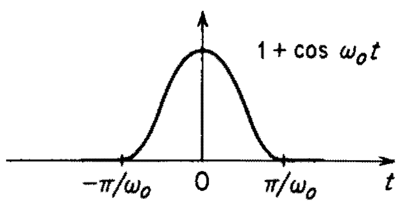
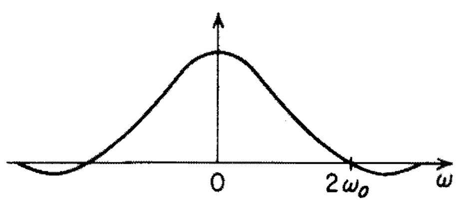
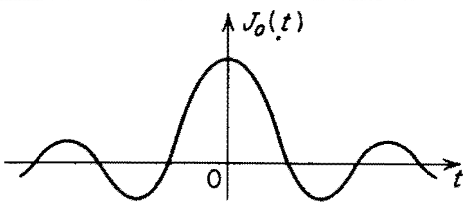
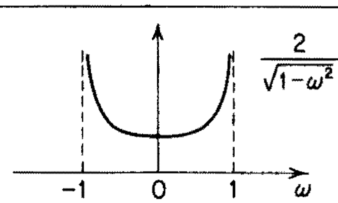
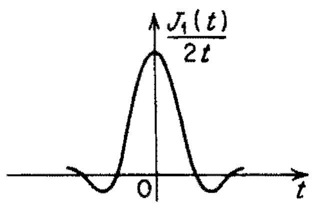
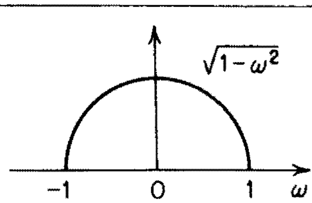
$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$	$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$
$p_a(t)$ 	$\frac{2\sin a\omega}{\omega}$ 
	$\frac{4\sin^2(a\omega/2)}{a\omega^2}$ 
$e^{-\alpha\|t\|}$ 	$\frac{2\alpha}{\alpha^2+\omega^2}$ 
e^{-at^2} 	$\sqrt{\frac{\pi}{a}}\,e^{-\omega^2/4a}$ 
$1+\cos\omega_0 t$ 	
$J_0(t)$ 	$\frac{2}{\sqrt{1-\omega^2}}$ 
$\frac{J_1(t)}{2t}$ 	$\sqrt{1-\omega^2}$ 

Table 1-2 Examples of Fourier transforms (continued)

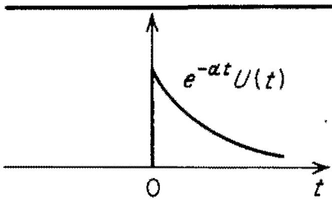
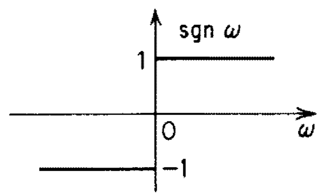
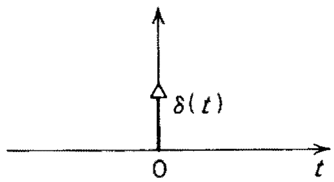
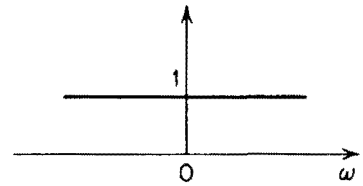
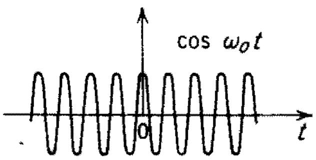
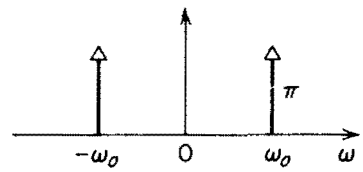
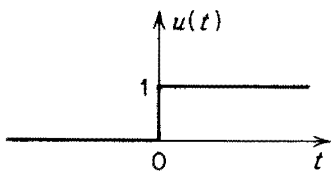
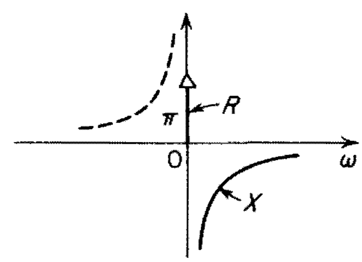
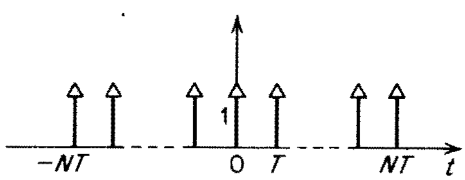
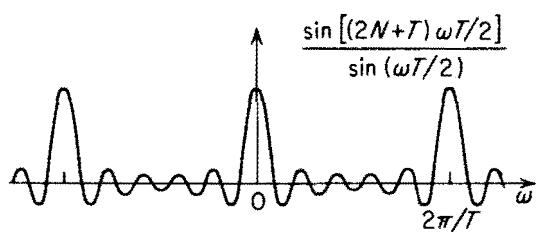
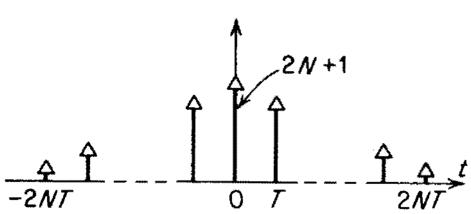
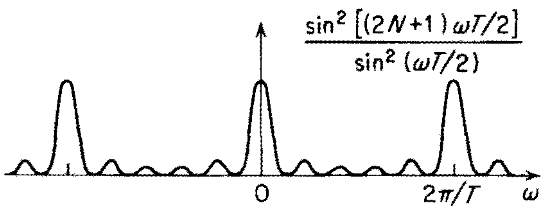
$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$	$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$
	$\frac{1}{\alpha + j\omega}$
$\frac{j}{\pi t}$	
$t^{\alpha}U(t) \qquad \alpha > -1$	$\frac{\Gamma(\alpha+1)}{\lvert\omega\rvert^{\alpha+1}} e^{\pm\frac{j\pi(\alpha+1)}{2}} \qquad \begin{matrix} - \text{ if } \omega > 0 \\ + \text{ if } \omega < 0 \end{matrix}$
$t^n e^{-\alpha t}U(t) \qquad \alpha > 0$	$\frac{n!}{(\alpha + j\omega)^{n+1}}$
$J_n(t)$	$\begin{cases} \frac{2\cos(n \arcsin \omega)}{\sqrt{1-\omega^2}} & n \text{ even } \lvert\omega\rvert < 1 \\ \frac{-2j\sin(n \arcsin \omega)}{\sqrt{1-\omega^2}} & n \text{ odd } \lvert\omega\rvert < 1 \\ 0 & \lvert\omega\rvert > 1 \end{cases}$
$\frac{J_n(t)}{t^n}$	$\frac{2(1-\omega^2)^{n-\frac{1}{2}}}{1\cdot 3\cdot 5\cdots(2n-1)} \quad \lvert\omega\rvert < 1$ $0 \quad \lvert\omega\rvert > 1$
$e^{j\alpha t^2}$	$\sqrt{\frac{\pi}{\alpha}}\, e^{j\pi/4} e^{-j\omega^2/4\alpha}$
$\cos \alpha t^2$ $\sin \alpha t^2$	$\sqrt{\frac{\pi}{\alpha}} \cos\left(\frac{\omega^2}{4\alpha} - \frac{\pi}{4}\right)$ $-\sqrt{\frac{\pi}{\alpha}} \sin\left(\frac{\omega^2}{4\alpha} - \frac{\pi}{4}\right)$
$e^{j\alpha t^2} \qquad 0 < t < T$ $0 \qquad$ otherwise	$\sqrt{\frac{\pi}{2\alpha}}\, e^{-j\omega^2/4\alpha}\left[\mathbf{F}\left(\sqrt{\alpha}\,T - \frac{\omega}{2\sqrt{\alpha}}\right) + \mathbf{F}\left(\frac{\omega}{2\sqrt{\alpha}}\right)\right]$ $\mathbf{F}(x) = \sqrt{\frac{2}{\pi}}\int_0^x e^{jy^2}\,dy$

Table 1-1 Transforms of singularity functions

$f(t) \leftrightarrow F(\omega)$	
	
	
	
	
	

$\delta(x)\delta(y) \Leftrightarrow 1$ $\qquad$ $\delta(x) \Leftrightarrow 2\pi\delta(v)$

$\varphi(x) \Leftrightarrow 2\pi\Phi(u)\delta(v)$ $\qquad$ $\varphi(x)\delta(y) \Leftrightarrow \Phi(u)$

Table 1-1 Hankel transform theorems

$$f(r) = \int_0^\infty w\bar{f}(w)J_o(rw)\,dw \overset{h}{\leftrightarrow} \bar{f}(w) = \int_0^\infty rf(r)J_o(wr)\,dr$$

$$f(\sqrt{x^2 + y^2}) \Leftrightarrow 2\pi\bar{f}(\sqrt{u^2 + v^2})$$

$\bar{f}(r)$	$f(w)$
$f(\alpha r)$	$\frac{1}{\alpha^2}\bar{f}\left(\frac{w}{\alpha}\right)$
$f''(r) + \frac{1}{r}f'(r)$	$-w^2\bar{f}(w)$
$f_1(r) ** f_2(r)$	$2\pi\bar{f}_1(w)\bar{f}_2(w)$

$$\int_0^\infty r\|f(r)\|^2\,dr = \int_0^\infty w\|\bar{f}(w)\|^2\,dw$$

$m_n = \int_0^\infty r^n f(r)\,dr$	$\bar{f}(w) = \sum_{n=0}^{\infty} \frac{(-1)^n m_{2n+1}}{(n!)^2 2^{2n}} w^{2n}$

$$\int_{-\infty}^{\infty} f(\sqrt{x^2 + y^2})\,dy \leftrightarrow 2\pi\bar{f}(u)$$

$$\int_0^\infty rf(r)e^{-j\omega r}\,dr = R_1(\omega) + jX_1(\omega)$$

$$\bar{f}(w) = \frac{2}{\pi}\int_0^{\pi/2} R_1(w\cos\theta)\,d\theta \qquad R_1(w) = w\int_0^{\pi/2} \bar{f}'(w\cos\theta)\,d\theta + \bar{f}(0)$$

Table 1-2 Examples of Hankel transforms

$f(r) = \int_0^\infty w\bar{f}(w)J_o(rw)\,dw$	$\overset{h}{\leftrightarrow}$ $\bar{f}(w) = \int_0^\infty rf(r)J_o(wr)\,dr$
$\frac{1}{r}$	$\frac{1}{w}$
$\delta(r-a)$	$aJ_o(aw)$
e^{-ar^2}	$\frac{1}{2a}e^{-w^2/4a}$
e^{jar^2}	$\frac{j}{2a}e^{-jw^2/4a}$
e^{-ar}	$\frac{a}{\sqrt{(a^2+w^2)^3}}$
$\frac{e^{-ar}}{r}$	$\frac{1}{\sqrt{a^2+w^2}}$
$\frac{\sin ar}{r}$	$\begin{cases} \frac{1}{\sqrt{w^2-a^2}} & w > a \\ 0 & w < a \end{cases}$
$\frac{J_n(r)}{r^n}$	$\begin{cases} \frac{(1-w^2)^{n-1}}{2^{n-1}(n-1)!} & w < 1 \\ 0 & w > 1 \end{cases}$
$\left.\begin{matrix} 1 & 0 < r < a \\ 0 & r > a \end{matrix}\right\}$	$\frac{aJ_1(aw)}{w}$
$\left.\begin{matrix} J_o(br) & 0 < r < a \\ 0 & r > a \end{matrix}\right\}$	$\frac{abJ_1(ab)J_o(aw) - awJ_o(ab)J_1(aw)}{b^2-w^2}$
$J_o^2(ar)$	$\begin{cases} \frac{2}{\pi w\sqrt{4a^2-w^2}} & w < 2a \\ 0 & w > 2a \end{cases}$
$\frac{J_o(ar)J_1(ar)}{r}$	$\begin{cases} \frac{1}{a\pi}\cos^{-1}\frac{w}{2a} & w < 2a \\ 0 & w > 2a \end{cases}$
$2\pi\frac{J_1^2(ar)}{r^2}$	$\begin{cases} 2\cos^{-1}\frac{w}{2a} - \frac{w}{a}\sqrt{1-\frac{w^2}{4a^2}} & w < 2a \\ 0 & w > 2a \end{cases}$