Dirac Delta Function

ME/EE 384N-8, Wave Phenomena, Spring 2024

Definition (sifting property):

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$$

Properties:

$$\delta(x) = 0\,, \quad x \neq 0$$
$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$
$$\delta(x) = \frac{d}{dx}H(x)$$
$$\delta(ax) = \frac{1}{|a|}\delta(x)$$
$$\delta[f(x)] = \sum_n \frac{\delta(x-x_n)}{|f'(x_n)|}\,, \quad f(x_n) = 0\,,\ n = 1, 2, ...$$
$$\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x-a)\,dx = (-1)^n f^{(n)}(a)\,, \quad f^{(n)}(x) = d^n f/dx^n$$
$$\delta(x)\delta(y) = \frac{\delta(\xi_1)\delta(\xi_2)}{J(\xi_1,\xi_2)}\,, \quad J = \left|\frac{\partial(x,y)}{\partial(\xi_1,\xi_2)}\right|\,, \quad dxdy = J(\xi_1,\xi_2)\,d\xi_1 d\xi_2$$

Functional representations:

$$\begin{aligned} \delta(x) &= \lim_{\epsilon\to 0} \frac{\mathrm{rect}(x/\epsilon)}{\epsilon} \\ &= \lim_{\epsilon\to 0} \frac{\sin(x/\epsilon)}{\pi x} \\ &= \lim_{\epsilon\to 0} \frac{e^{-x^2/\epsilon^2}}{\epsilon\sqrt{\pi}} \\ &= \lim_{\epsilon\to 0} \frac{\epsilon/\pi}{x^2+\epsilon^2} \\ \delta(x-x_0) &= \sum_n \phi_n(x)\phi_n(x_0)\,, \quad \{\phi_n\} \text{ a complete orthonormal set} \end{aligned}$$

Integral representations:

1D: $$\int_{-\infty}^{\infty} e^{ik_x(x-x_0)}\,dk_x = 2\pi\delta(x-x_0)$$

2D: $$\int_0^{\infty} J_\alpha(\kappa\rho)J_\alpha(\kappa\rho_0)\,\kappa\,d\kappa = \frac{1}{\rho}\delta(\rho-\rho_0)\,, \quad \alpha > -\tfrac{1}{2}$$

3D: $$\int_0^{\infty} j_\alpha(kr)j_\alpha(kr_0)\,k^2\,dk = \frac{\pi}{2r^2}\delta(r-r_0)\,, \quad \alpha > -1$$

Related transforms:

$$
\begin{aligned}
\mathcal{F}_x\{e^{ik_0x}\} &= 2\pi\delta(k_x - k_0) \\
\mathcal{F}_x\{\cos k_0x\} &= \pi[\delta(k_x + k_0) + \delta(k_x - k_0)] \\
\mathcal{F}_x\{\sin k_0x\} &= i\pi[\delta(k_x + k_0) - \delta(k_x - k_0)]
\end{aligned}
$$

2D delta functions:
Cartesian coordinates

$$
\begin{aligned}
\delta(\boldsymbol{\rho} - \boldsymbol{\rho}_0) &= \delta(x - x_0)\delta(y - y_0) \\
&= \delta(x)\delta(y)\,, \quad \boldsymbol{\rho}_0 = 0
\end{aligned}
$$

Polar coordinates

$$
\begin{aligned}
\delta(\boldsymbol{\rho} - \boldsymbol{\rho}_0) &= \frac{\delta(\rho - \rho_0)}{\rho}\delta(\phi - \phi_0) \\
&= \frac{\delta(\rho)}{2\pi\rho}\,, \quad \boldsymbol{\rho}_0 = 0
\end{aligned}
$$

The convention used above is $\int_0^\infty \delta(\rho)\,d\rho = 1$ to satisfy the requirement that the integral over area in polar coordinates be unity for $\boldsymbol{\rho}_0 = 0$: $\int_0^\infty \int_0^{2\pi} \delta(\boldsymbol{\rho})\,\rho d\rho\,d\phi = 1$. Some authors use the convention $\int_0^\infty \delta(\rho)\,d\rho = \frac{1}{2}$, which then requires $\delta(\boldsymbol{\rho}) = \delta(\rho)/\pi\rho$ for the integral to be unity.

3D delta functions:
Cartesian coordinates

$$
\begin{aligned}
\delta(\mathbf{r} - \mathbf{r}_0) &= \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \\
&= \delta(x)\delta(y)\delta(z)\,, \quad \mathbf{r}_0 = 0
\end{aligned}
$$

Polar coordinates

$$
\begin{aligned}
\delta(\mathbf{r} - \mathbf{r}_0) &= \frac{\delta(\rho - \rho_0)}{\rho}\delta(\phi - \phi_0)\delta(z - z_0) \\
&= \frac{\delta(\rho)}{2\pi\rho}\delta(z)\,, \quad \mathbf{r}_0 = 0
\end{aligned}
$$

Spherical coordinates

$$
\begin{aligned}
\delta(\mathbf{r} - \mathbf{r}_0) &= \frac{\delta(r - r_0)}{r^2}\frac{\delta(\theta - \theta_0)}{\sin\theta}\delta(\phi - \phi_0) \\
&= \frac{\delta(r)}{4\pi r^2}\,, \quad \mathbf{r}_0 = 0
\end{aligned}
$$

As in polar coordinates the convention $\int_0^\infty \delta(r)\,dr = 1$ is used.